

REVIEW SESSION ON COMPLEX ANALYSIS

MATH 241

These problems are definitely above the average level, I want to make it a little hard to motivate you to review, and design a lot of traps to help you discover the possible mistakes. So do not be anxious too much about it. Work hard and take it easy!

1. Write down the Maclaurin expansion of $\frac{1}{1+e^z}$ up to degree 2. What is the radius of convergence of its Maclaurin expansion?

Proof. If you want to mimic geometric series and get $1 - e^z + e^{2z} - e^{3z} + \dots$, and plug in the Maclaurin series of e^z , you will have $1 - 1 + 1 - 1 + 1 - 1 + \dots$ as the constant term, which does not converge at all. So the safest way is to differentiate and calculate the first and second order derivatives.

The alternative way is to write it as

$$\frac{1}{1 + 1 + z + \frac{z^2}{2!} + \dots} = \frac{1}{2 + z + \frac{z^2}{2!} + \dots} = \frac{1}{2} \frac{1}{1 + \frac{1}{2}z + \frac{1}{4}z^2 + HOT} = \frac{1}{2} (1 - (\frac{1}{2}z + \frac{1}{4}z^2 + HOT) + (\frac{1}{2}z + \frac{1}{4}z^2 + HOT)^2 + HOT)$$

So after simplification we get

$$\frac{1}{2} - \frac{1}{4}z + HOT$$

as the Maclaurin expansion up to degree 2. □

2. Sketch the domain for z on which e^{2z} has a positive real part.

Proof. $e^{2z} = e^{2x}(\cos 2y + i \sin 2y)$. e^{2x} is always positive, so we want $\cos(2y) > 0$. This means $2y \in (-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi)$, or equivalently $y \in (-\frac{\pi}{4} + n\pi, \frac{\pi}{4} + 2n\pi)$. □

3. For the vector field $V = (x + 2y, y)$, what are the circulation and the net flux around the unit circle $\{z \mid |z| = 1\}$?

Proof. Write as $f(z) = x + 2y + iy$. Then $\oint_C \overline{f(z)} dz$ is what we want. Use the usual parametrization $x = \cos t, y = \sin t$, we get $\int_0^{2\pi} (\cos t + 2 \sin t - i \sin t)(-\sin t + i \cos t) dt$. Simplify we get $\int_0^{2\pi} (-2 \sin^2 t) dt + i \int_0^{2\pi} 1 + 2 \sin t \cos t dt$. By double angle formula, $\int_0^{2\pi} (-2 \sin^2 t) dt = \int_0^{2\pi} -1 + \cos(2t) dt = -2\pi$, and $\int_0^{2\pi} 1 + 2 \sin t \cos t dt = 2\pi + \int_0^{2\pi} \sin 2t dt = 2\pi$. So $\oint_C \overline{f(z)} dz = -2\pi + 2\pi i$, so circulation is the real part -2π , and net flux is the imaginary part 2π . □

4. Calculate P.V. $\int_{-\infty}^{\infty} \frac{1}{x^4 + x^2 + 1} dx$.

Proof. The zeroes of $x^4 + x^2 + 1$ are the square roots of the solution to the quadratic equation $t^2 + t + 1 = 0$, so $x^2 = \frac{-1 \pm \sqrt{3}i}{2} = e^{\frac{2\pi}{3}i}, e^{-\frac{2\pi}{3}i}$. So $x = e^{\frac{\pi}{3}i}, e^{\frac{2\pi}{3}i}, e^{-\frac{\pi}{3}i}, e^{-\frac{2\pi}{3}i}$. The poles on the upper half plane are $e^{\frac{2\pi}{3}i}, e^{\frac{\pi}{3}i}$. To calculate the residues there, we need $\frac{1}{(x^4+x^2+1)'}$'s evaluation since they are all simple poles. We get $\frac{1}{4x^3+2x}|_{e^{\frac{2\pi}{3}i}} = \frac{1}{4e^{2\pi i}+2e^{\frac{2\pi}{3}i}} = \frac{1}{4+2(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} = \frac{1}{3+\sqrt{3}i} = \frac{3-\sqrt{3}i}{12}$. That was $\text{Res}(f(z), e^{\frac{2\pi}{3}i})$. Similarly, $\frac{1}{4x^3+2x}|_{e^{\frac{\pi}{3}i}} = \frac{1}{4e^{\pi i}+2e^{\frac{\pi}{3}i}} = \frac{1}{-4+2(\frac{1}{2}+\frac{\sqrt{3}}{2}i)} = \frac{1}{-3+\sqrt{3}i} = \frac{-3-\sqrt{3}i}{12}$. That was $\text{Res}(f(z), e^{\frac{\pi}{3}i})$. Therefore the answer is $2\pi i(\text{Res}(f(z), e^{\frac{2\pi}{3}i}) + \text{Res}(f(z), e^{\frac{\pi}{3}i})) = 2\pi i(\frac{3-\sqrt{3}i}{12} + \frac{-3-\sqrt{3}i}{12}) = \pi$. \square

5. Find all the solutions to $z^3 + (z+2)^3 = 0$.

Proof. This equation is equivalent to $(\frac{z}{z+2})^3 = -1 = e^{\pi i}$. So $\frac{z}{z+2} = e^{\frac{\pi}{3}i}, e^{\pi i}, e^{\frac{5\pi}{3}i}$, or, more explicitly, $\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i$. They solve out $z = -1 + \sqrt{3}i, -1, -1 - \sqrt{3}i$. \square

6. Let C be the ellipse $4x^2 + y^2 = 9$, oriented clockwise. Calculate $\oint_C \frac{\sin z}{z^2+4} dz$.

Proof. There are two poles, $z = \pm 2i$, both inside the ellipse since they correspond to point $(0, \pm 2)$, while the intersections of the ellipse with the y-axis are ± 3 . They are both simple poles, so the residues are $\frac{\sin z}{2z}|_{z=\pm 2i}$, i.e., $\frac{\sin(2i)}{4i} = \frac{e^2 - e^{-2}}{8}$ and $\frac{\sin(-2i)}{-4i} = \frac{e^2 - e^{-2}}{8}$. Notice the orientation is clockwise, so the answer is $-2\pi i(\text{Res}(f(z), 2i) + \text{Res}(f(z), -2i)) = \frac{e^2 - e^{-2}}{2}\pi i$. \square

7. Write down the Laurent expansion of $\frac{z+1}{(z-5)^2}$ from degree -4 to $+4$ on: (1) $\{z|0 < |z-5| < 10\}$; (2) $\{z|6 < |z| < 8\}$.

Proof. (1) $\frac{z+1}{(z-5)^2} = \frac{(z-5)+4}{(z-5)^2} = \frac{4}{(z-5)^2} + \frac{1}{z-5}$.

(2) First partition of fraction, get $\frac{4}{(z-5)^2} + \frac{1}{z-5}$. The singularity $z = 5$ is inside the annulus, so

$$\frac{4}{(z-5)^2} = 4 \cdot \frac{1}{z^2} \frac{z^2}{(z-5)^2} = 4 \cdot \frac{1}{z^2} \frac{1}{(1-\frac{5}{z})^2} = 4 \cdot \frac{1}{z^2} (1 + \frac{(-2)}{1!}(-\frac{5}{z}) + \frac{(-2)(-3)}{2!}(-\frac{5}{z})^2 + \text{LOT})$$

Here "LOT" stands for "lower order terms". So simplify to get

$$\frac{4}{z^2} + \frac{40}{z^3} + \frac{300}{z^4} + \text{LOT}$$

\square

8. What is the set where $f(x+iy) = (x^2 - 2y) + i(\frac{2}{3}x^3 + xy)$ is differentiable? analytic?

Proof. Let $u = x^2 - 2y, v = \frac{2}{3}x^3 + xy$. $\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = x$, so if Cauchy-Riemann equation holds, $2x = x$, so $x = 0$. $\frac{\partial u}{\partial y} = -2, \frac{\partial v}{\partial x} = 2x^2 + y$, so if Cauchy-Riemann equation holds, $2x^2 + y = -2$. We know $x = 0$, so $y = -2$. So the function is differentiable only at $z = -2i$. Since it does not have any interior point, the function is nowhere analytic. \square